References

[1] Dallison, K. J., "Stress analysis of circular frames in a non-tapering fuselage," J. Roy. Aeron. Soc. **57**, 151–176 (1953).

[2] Argyris, J. H., "Energy theorems and structural analysis," Aircraft Eng. (October 1954 to May 1955).

[3] Asplund, S. O., *A Unified Analysis of Indeterminate Structures* (Transactions of Chalmers University of Technology, Gothenburg, Sweden, (1961).

[4] Asplund, S. O., *Structural Mechanics* (Chalmers University of Technology, Gothenburg, Sweden, 1963), Part II.

[5] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and deflection analysis of complex structures," J. Aeronaut. Sci. 805–823 (1956).

[6] Jones, R. E., "A generalization of the direct-stiffness method of structural analysis," AIAA J. **2**, 821–826 (1964).

A Solution of the Partial Differential Equation Governing Vibration of a Rod

SING-CHIH TANG*

Ford Motor Company, Dearborn, Mich.

Introduction

IN this paper, a numerical solution by the method of characteristics is presented for the transient response of a semi-infinite circular rod where one end is restrained radially and subjected to a step-axial pressure. The equation of motion for such a rod is based upon a one-dimensional theory of compressional waves including effects of the radial inertia and the radial shear derived by Mindlin and Herrmann.[1] It is well known that the result from this theory matches quite closely that of the exact theory due to Pochhammer. Several authors[2,3] have solved this problem by the method of integral transforms. Since this kind of equation of motion is composed of a number of simultaneous equations, the characteristic equation is a high-order polynomial, and the inversion integrals are complicated. Numerical methods must be used to evaluate those complicated inversion integrals. This suggests a direct numerical method to attack the problem. Since the system of equations is hyperbolic[4] and contains only one spatial variable, a simple numerical procedure by the method of characteristics can be used. Leonard and Budiansky[5] and Plass[6] applied the method of characteristics to the solution of the beam vibration problem by the Timoshenko theory. The author[7] applied it to solve the transient response of a thin-walled cylindrical tube under internal moving pressure.

Equation and Method of Solution

The homogeneous equations of motion and the relationship between the axial stress and displacement according to the Mindlin-Herrmann theory are as follows:

$$\kappa^2 a^2\mu(\partial^2 u/\partial x^2) - 8\kappa_1^2(\lambda + \mu)u - 4\lambda\kappa_1 a(\partial w/\partial x) = \rho a^2(\partial^2 u/\partial t^2) \quad (1)$$

$$2\lambda\kappa_1 a(\partial u/\partial x) + (\lambda + 2\mu)a^2(\partial^2 w/\partial x^2) = \rho a^2(\partial^2 w/\partial t^2) \quad (2)$$

$$2P_x = 2\kappa_1 a\lambda u + (\lambda + 2\mu)a^2(\partial w/\partial x) \quad (3)$$

in which u, w are the radial and axial displacements, respectively; P_x is the resultant axial stress on a section perpendicular to the axis of the rod; a is the radius of the rod; λ, μ are Lamé constants; ρ is the density of the rod; x is the axial distance; and t is the time variable. κ and κ_1 are correction factors determined as follows: κ^2 is the root of the equation

$$4[(1 - \zeta\kappa^2)(1 - \kappa^2)]^{1/2} = (2 - \kappa^2)^2 \qquad 0 < \kappa < 1 \quad (4)$$

where $\zeta = (1 - 2\nu)/2(1 - \nu)$, and ν is Poisson's ratio,

$$\kappa_1^2 = \eta^2(\lambda + 2\mu)/8(\lambda + \mu) \quad (5)$$

where η is the lowest root of the equation

$$[(\lambda + 2\mu)/2\mu]\eta J_0(\eta) = J_1(\eta) \quad (6)$$

and J_0, J_1 are Bessel functions of order zero and one, respectively.

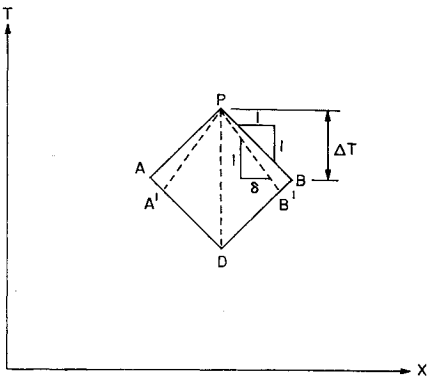

Fig. 1 Typical element of grid.

It is assumed that there is no motion initially for the rod and that a step axial pressure P_0 is applied for $t > 0$ at the end $x = 0$ while the rod is restrained radially at this end. The stress and the displacement at the end $x = \infty$ are not affected.

The system of equations (1) and (2) is hyperbolic,[4] and its characteristics are given by $dt/dx = \pm 1/c_1$; $dt/dx = \pm 1/c_2$, where $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$, the dilatational wave velocity; and $c_2 = \kappa(\mu/\rho)^{1/2}$, the modified shear-wave velocity. If four dependent variables, $U_x = \partial U/\partial X$, $U_T = \partial U/\partial T$, $W_x = \partial W/\partial X$, and $W_T = \partial W/\partial T$, in which the dimensionless variables $U = u/a$, $W = w/a$, $X = x/a$, and $T = c_1 t/a$ are introduced, Eqs. (1) and (2) can be transformed into a set of four first-order equations in dimensionless form defined along the characteristics as follows.

Along the characteristics $dT/dX = \pm 1$,

$$dW_x \mp dW_T \pm \beta U_x dT = 0 \quad (7a, b)$$

Along the characteristics $dT/dX = \pm 1/\delta$, where $\delta = c_2/c_1$,

$$\delta dU_x \mp dU_T \mp (\alpha U + 2\beta W_x)dT = 0 \quad (8a, b)$$

In Eqs. (8a, b) U† can be expressed by the integral

$$U = U_0 + \int_{T_0}^{T} U_T dT \quad (9)$$

where U_0 is the radial displacement of the same point at time T_0. The constants in Eqs. (7a, b) and (8a, b) are

$$\alpha = 4\kappa_1^2/(1 - \nu) \qquad \beta = 2\kappa_1[\nu/(1 - \nu)]$$

Equation (3) can also be transformed to the dimensionless form by

$$P = \beta U + W_x \quad (10)$$

where

$$P = [2/a^2(\lambda + 2\mu)]P_x$$

There are two kinds of waves in this elastic system: the dilatational wave is propagated with dimensionless velocities

Received November 30, 1964; revision received March 17, 1965.

* Research Scientist Engineering Mechanics Department, Applied Research Office, Engineering and Research Staff.

† U may be treated as the fifth dependent variable; then the fifth equation is $U_T = \partial U/\partial T$. The final result will be the same.

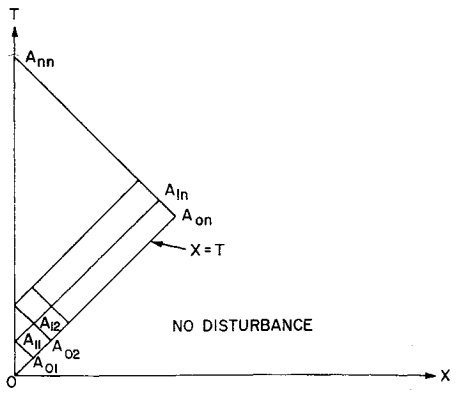

Fig. 2 Triangular region used in computation.

± 1 along the lines $dT/dX = \pm 1$, whereas the modified shear wave is propagated with dimensionless velocities $\pm\delta$ along the lines $dT/dX = \pm 1/\delta$. Since the dilatational wave velocity is larger than that of the modified shear wave and the rod is quiescent initially, there is no disturbance before the dilatational wave, excited by the step pressure at the end $x = 0$, arrives.

A finite-difference procedure for the numerical integration of Eqs. (7–9) has been performed on the grid formed by the characteristics of the dilatational wave family. A typical element‡ of the grid is shown in Fig. 1. If U_x, U_T, W_x, W_T, and U are given at D, A, and B, those values at P can be found from Eqs. (7–9) by the numerical procedure. By successive use of this procedure, all values at grid points $A_{01}, A_{02}, \ldots, A_{0n}$; $A_{11}, A_{12}, \ldots, A_{1n}$; $\ldots$; A_{nn} in Fig. 2 can be evaluated.

The only discontinuity of the boundary and initial conditions is at the origin ($X = T = 0$), since the input axial pressure at $X = 0$ is a step function. This discontinuity is propagated along the characteristic $X = T$ and makes W_x and W_T among the five dependent variables W_x, W_T, U_x, U_T, and U discontinuous across it. From Eq. (10), since U is continuous, the dimensionless magnitude of the jump for W_x at the origin is

$$P_D = 2P_0/a^2(\lambda + 2\mu) \tag{11}$$

By establishing Eqs. (7a, b) along BP and AP, respectively, where B, P, and A are arbitrarily close to the line $X = T$ as shown in Fig. 3, one knows that W_x, as well as $-W_T$, has a discontinuity of magnitude P_D across the characteristic $X = T$ at the origin, and it is propagated along $X = T$ without change of magnitude.

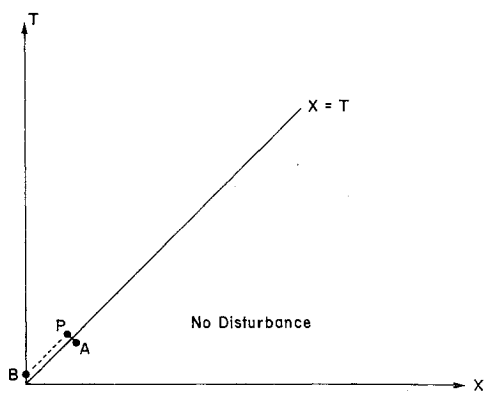

Fig. 3 Line of discontinuity $X = T$.

‡ At the boundary, the element becomes a triangle PBD. If those values at B, D, and two of the five values at P are given, the three others at P can be determined by Eqs. (7b, 8b, and 9).

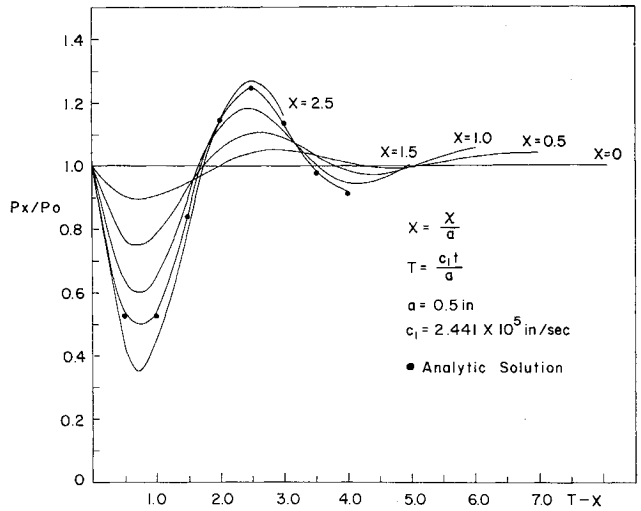

Fig. 4 Axial-rod stress with time at various stations.

Numerical Example

A numerical example based upon this method is given below for the following data:

$$\nu = 0.325 \qquad \varkappa_1 = 0.767 \qquad \varkappa^2 = 0.8667$$
$$a = 0.5 \text{ in.} \qquad \Delta T = 0.01$$

where ΔT is the dimensionless time interval to be taken in the numerical integration. The result is shown in Fig. 4. At each station $X = 0.5$, 1.0, 1.5, 2.0, and 2.5, the axial-rod stress ratio P_x/P_0 is plotted vs time duration $(T - X)$ after the dilatational wave has arrived. An analytic solution from Ref. 2 is also plotted in the same figure as a check of the direct numerical solution.

Discussion

The numerical technique used in this problem is quite simple. The grid is formed by characteristics that have the smallest magnitude in slope. For this reason, the numerical method is automatically stable. In the process of computation, W_x, W_T, U_x, U_T, and U, i.e., stresses and displacements, can be found at the same time; the analytic methods do not have this advantage.

This method can be extended to all transient vibration problems of the continued system as long as the governing equation of motion is hyperbolic. One of the most important examples is that of a semi-infinite cylindrical shell with consideration of the rotatory inertia and shear deformation.[7] The nonhomogeneous equation can be treated without difficulty by means of this method.

The only disadvantage of this method is the long computer time required, even on a large-scale digital computer, such as the IBM 7090. In this example, the dispersion is very obvious within a short time, so that computing time is not long. For the same problem on a thin-walled shell, a longer time must be allowed if the apparent dispersion is to be obtained.

References

[1] Mindlin, R. D. and Herrmann, G., "A one-dimensional theory of compressional waves in an elastic rod," *Proceedings of the First U. S. National Congress of Applied Mechanics, June, 1951* (American Society of Mechanical Engineers, New York, 1952), pp. 187–191.

[2] Miklowitz, J., "The propagation of compressional waves in a dispersive elastic rod, Part I—Result from the theory," J. Appl. Mech. **24**, 231–239 (1957).

[3] Kaul, R. K., "On the propagation of pressure pulses in circular elastic rods," Z. Angen. Math. Phys. **14**, 704–712 (1963).

[4] Petrovsky, I. G., *Lectures on Partial Differential Equations*, transl. by A. Shenitzer (Interscience Publishers, Inc., New York, 1954), 1st English ed., Chap. I.

[5] Leonard, R. W. and Budiansky, B., "On traveling waves in beams," NACA TN 2874 (1953).

[6] Plass, H. J., Jr., "Some solutions of the Timoshenko beam equation for short pulse-type loading," J. Appl. Mech. **25**, 379–385 (1958).

[7] Tang, S. C., "Dynamic response of a thin-walled cylindrical tube under internal moving pressure," Ph. D. Thesis, Univ. of Michigan (1963).

Altitude-Velocity Charts for Imperfect Air

CLARK H. LEWIS* AND E. G. BURGESS III†
ARO, Inc., Arnold Air Force Station, Tenn.

NUMEROUS authors have computed normal shock-wave parameters based on perfect gas‡ thermodynamic properties and various "standard" atmospheres (e.g., see Feldman,[1] Wittliff and Curtis,[2] and Marrone[3]). At sufficiently low pressures, except for changes in standard atmosphere conditions, those results should be sufficiently accurate for most engineering purposes. However, at high pressures (or more correctly at high densities), the effects of intermolecular forces can become important and the thermodynamic properties are affected.

Lewis and Burgess[4] recently computed freestream stagnation conditions, conditions behind a normal shock, and nor-

Table 1 Ranges of conditions considered in altitude-velocity calculations

Altitude-velocity ranges			
	Model atmosphere	Altitude range, kft	Velocity range, kft/sec
Wittliff and Curtis	1959	0–300	2–26
Marrone	1959	0–300	26–50
Lewis and Burgess	1962	10–400	6–50

Gasdynamic properties[a]	
Wittliff and Curtis[b] and Marrone[b]:	ρ_s/ρ_1, p_s/p_1, T_s/T_1, Z_s, H_s/H_1, H_0'/H_1, p_0'/p_1, T_0'/T_1, γ_e[c]
Lewis and Burgess:	H_0/H_1, p_0/p_1, ρ_0/ρ_1, T_0/T_1, a_0/a_a, Z_0, $\gamma_E, {}_{0s}$[d]
	H_s/H_1, p_s/p_1, ρ_s/ρ_1, T_s/T_1, S_s/R, Re_1/ft, $\gamma_{E,s}$
	$\dot{q}(r_n)^{1/2}$, p_0'/p_1, ρ_0'/ρ_1, T_0'/T_1, a_0'/a_a, Z_0', $\gamma_{E,0'}$

[a] Subscript 1 denotes freestream, s behind normal shock, 0 freestream stagnation, 0′ normal shock stagnation, and a at 273.15°K and 1 atm.

[b] Gas composition behind the normal shock also given based on 14 specie model.

[c] $\gamma_e = 2M_1^{-2}(\rho_1/\rho_s - 1)^{-1} - 1$.

[d] $\gamma_E = (\partial \ln p/\partial \ln \rho)_S$.

Received December 10, 1964. This work was sponsored by the Arnold Engineering Development Center (AEDC), Air Force Systems Command, U. S. Air Force, under Contract No. AF 40(600)-1000 with ARO, Inc., Operating Contractor, AEDC.

* Engineer, Hypervelocity Branch, von Karman Gas Dynamics Facility. Member AIAA.

† Member, Scientific Computing Branch, Engineering Support Facility.

‡ The ideal gas obeys $p = \rho RT$, $h = C_pT$, and $\gamma = C_p/C_v$ = const. A perfect gas will denote one obeying $p = Z\rho RT$, which includes dissociation and ionization neglecting intermolecular effects. An imperfect gas obeys $p = Z\rho RT$, which includes dissociation, ionization, and intermolecular forces. Local thermodynamic [i.e., thermal, mechanical (pressure), and chemical] equilibrium is assumed to exist for all conditions.

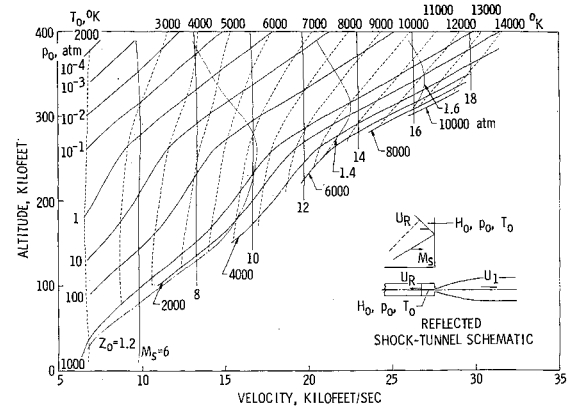

Fig. 1 Equilibrium freestream stagnation conditions necessary for flight duplication.

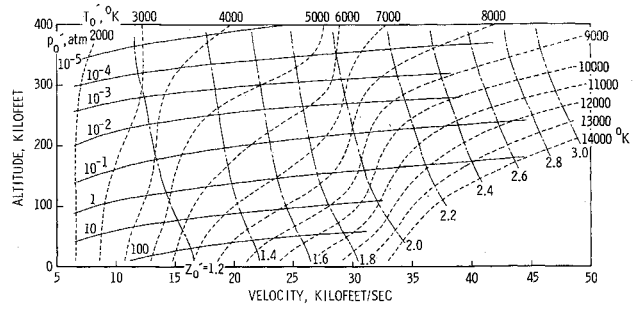

Fig. 2 Normal shock stagnation conditions at flight duplication.

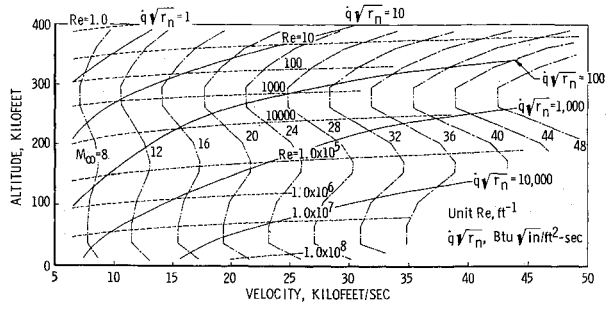

Fig. 3 Freestream Mach number, unit Reynolds number, and Fay-Riddell stagnation heat-transfer rate at flight duplication.

Table 2 Comparison of Air Research and Development Command (1959) and U S. Standard (1962) model atmospheres[a]

Geometric alt., kft	p_1	ρ_1	T_1	H_1/R^b	$a_1{}^b$
0	...	...	...	−0.4	...
70	−0.06	0.52	−0.57	−1.06	−0.32
100	−0.8	−3.16	2.50	2.00	1.21
120	2.44	−1.58	4.12	3.62	2.00
160	9.41	4.77	4.45	5.98	2.15
170	11.2	6.45	4.34	3.98	2.15
200	14.1	16.09	−1.75	−2.22	−0.92
220	11.15	15.5	−3.76	−4.21	−1.93
250	2.74	10.2	−6.74	−7.16	−3.45
280	−11.25	−2.96	−8.29	−8.70	−4.27

[a] Data shown were computed from percent difference = 100 [x(1959) − x(1962)]/x(1962).

[b] H(1959) and a(1959) were obtained from Wittliff and Curtis.[2]